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Flux across nonsmooth boundaries and fractal Gauss/Green/Stokes' theorems

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Abstract. By replacing the parametrization of a domain with polyhedral approximations we give optimal extensions of theorems of Gauss, Green and Stokes'. Permitted domains of integration range from smooth submanifolds to structures that may not be locally Euclidean and have no tangent vectors defined anywhere. One may still calculate divergence and curl over a domain, and flux across its boundary which itself may have no normal vectors defined anywhere.

Introduction

The real numbers $\mathbb{R}$ are a completion of the rationals via the Euclidean metric. Continuity properties of real numbers, relative to the Euclidean metric, are at the heart of a real analysis. Similarly, one may consider the vector space of p-dimensional simplicial chains $\sum_{i=1}^{k} a_i \sigma_i$ in $\mathbb{R}^n$ and their completion w.r.t. a norm†. The Banach space obtained on completion has limit points that can be written as conditionally convergent series of simplicial chains,

$$A = \sum_{i=1}^{\infty} a_i \sigma_i$$

called *chainlets*. In [H2] the author defined a family of norms giving geometric meaning to these infinite series of weighted simplexes and thus to chainlets. (See section 1, below.)

The integral of a smooth form ω over a chainlet $\sum_{i=1}^{\infty} a_i \sigma_i$ is defined using term-by-term integration

$$\int_{\sum_{i=1}^{\infty} a_i \sigma_i} \omega = \sum_{i=1}^{\infty} \int_{a_i \sigma_i} \omega.$$

Examples of chainlets include smooth submanifolds, fractals, vector fields, Dirac delta masses, Cantor sets, and stable manifolds and the theory shows how they all fit together continuously into Banach spaces. Some examples are further described in sections 2 and 3. The author [H4] has shown that a large subspace of distributions and currents corresponds to chainlets. This not only provides a large source of examples, but can be used to show that a number of generalizations of classical results are optimal. While distributions and currents are defined abstractly as linear functionals on functions and differential forms, respectively, we emphasize that chainlets have concrete geometric definition.

† One may also work with ambient spaces of Riemannian manifolds. (See [H5].)

The classical theory of differential manifolds relies heavily on results of linear algebra of tangent spaces. Much of the work involves taking partitions of unity or checking coherence in the overlap maps. These techniques are not necessary or even valid for the theory of chainlets which are not assumed to be locally Euclidean and thus may have no tangent spaces. Instead, one replaces linear algebra on tangent spaces with analysis on simplexes. Parametrization is replaced with simplicial approximation. Partitions of unity are replaced with algebraic sums of chains. The unit normal bundle of a smooth submanifold B is replaced with $*B$, the chainlet that is the geometric Hodge * of B. (See section 3.) Because of the continuity of fundamental operators, results such as Gauss/Green/Stokes' theorems for simplexes carry immediately over to chainlets. Other important results of calculus, algebraic topology, differential topology, and measure theory extend to chainlets providing a common language for these theories.

The methods of this paper are distinct from those of geometric measure theory (GMT). In GMT, one begins with spaces of differential forms and defines currents abstractly as dual operators on forms. Inspired by the approach of Whitney [W] in his book Geometric Integration Theory, we start with spaces of domains defined geometrically—chainlets—and in [H3] prove that linear operators on them correspond uniquely to differential forms. In [H4] we complete the picture by finding topologies on forms so that the resulting currents correspond uniquely to chainlets. This leads to geometric methods for the study of distributions and currents as well as many new examples. Our Gauss–Green theorems are more general than those proved by Federer and de Giorgi [F, deG]. They worked with boundaries of n-dimensional domains in $\mathbb{R}^n$ that are rectifiable and used the fact that rectifiable boundaries have measure-theoretic normals defined almost everywhere. The domains of our extension may have unrectifiable boundaries and may be of any codimension. Examples include compact surfaces with infinitely long boundaries in three-space Our results also go beyond those of [H-N] where, again, the domains are top dimensional and there is no geometric Hodge star operator.

1. Dipoles and norms

An *oriented p-simplex* in $\mathbb{R}^n$ is the oriented convex hull of $p + 1$ points in $\mathbb{R}^n$. We assume all simplexes are oriented henceforth. A *simplicial chain* in $\mathbb{R}^n$ is a formal sum of simplexes in $\mathbb{R}^n$ with real coefficients. We may assume that integration of smooth forms is defined over simplexes and thus over simplicial chains, and that Stokes' theorem is valid for simplicial chains $\int_{\partial S} \omega = \int_S \mathrm{d}\omega$. The mass of a simplicial chain $\sum_i a_i \sigma_i$ is simply the weighted sum of masses of the simplexes $|S|_0 = \sum_i |a_i| m(\sigma_i)$ where m denotes p-dimensional Lebesgue measure. If $v \in \mathbb{R}^n$ is a vector let $|v|$ denotes its length. If σ is a p-simplex in $\mathbb{R}^n$ and $v \in \mathbb{R}^n$, define $T_v\sigma$ as the translate of σ by v. Its orientation is naturally induced from the orientation given on σ.

The mass of simplicial chains does not naturally measure geometric continuity. For example, the simplicial chain $\sigma - T_{tv}\sigma$ has mass that is twice the mass of σ unless $t = 0$. This problem is partially circumvented with polyhedral chains.

Polyhedral chains are equivalence classes of simplicial chains satisfying

$$S \sim T \iff \int_S \omega = \int_T \omega$$

for all smooth ω. Write $A = [S]$ and define $\int_A \omega = \int_S \omega$. For example, $-\sigma$ is identified with the same simplex as σ but with the opposite orientation. This definition takes into account overlapping simplexes with the opposite sign. The region of overlap is cancelled. Polyhedral chains have naturally defined mass

$$|A|_0 = \inf\{|S|_0 : A = [S]\}.$$

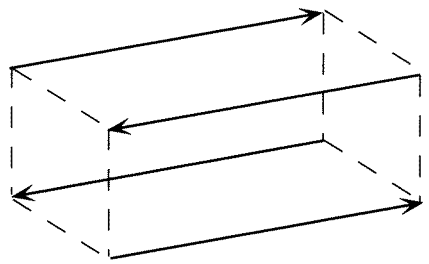

Figure 1. A simple two-dipole, or quadrupole.

The mass of an n-dimensional polyhedral chain $A = [\sigma - T_{tv}\sigma]$ in $\mathbb{R}^n$ *is* a continuous function of t. Because of the cancellation of overlapping oppositely oriented simplexes, the mass tends to zero as $t \to 0$.

Unfortunately, mass alone is not enough to conclude that a chain and its nearby translate are close to each other. Consider a two-simplex σ in $\mathbb{R}^3$ and v a vector not in the plane of σ. Then σ and $T_{tv}\sigma$ are disjoint if $t \neq 0$. The mass of $[\sigma - T_{tv}\sigma]$ is again exactly twice the mass of σ, until $t = 0$, at which point the mass becomes 0. We need finer norms than mass to be sensitive to geometric continuity.

Dipoles. A *simple p-dimensional zero-dipole* in $\mathbb{R}^n$ is defined to be a p-simplex σ^0 with diameter $\leqslant 1$. A *simple p-dimensional one-dipole* is a p-chain of the form

$$\sigma^1 = \sigma^0 - T_{v_1}\sigma^0$$

where $|v_1| \leqslant 1$ and σ^0 is disjoint from $T_{v_1}\sigma^0$. We inductively define simple j-dipoles. Given a vector v_j with $|v_j| \leqslant 1$ and a simple $(j-1)$-dipole σ^{j-1} disjoint from $T_{v_j}\sigma^{j-1}$, define the *simple p-dimensional j-dipole σ^j* as the simplicial p-chain

$$\sigma^j = \sigma^{j-1} - T_{v_j}\sigma^{j-1}.$$

Thus σ^j is generated by vectors $v_1, \ldots, v_j$, each with norm $\leqslant 1$, and a simplex σ^0, where all translations of σ^0 through the vectors v_i are disjoint. (See figure 1.)

A *j-dipole in* $\mathbb{R}^n$, is a simplicial chain of simple j-dipoles, $D^j = [\sum_{i=1}^k a_i\sigma_i^j]$ with real coefficients a_i.

Dipole mass. Given a simple j-dipole σ^j, generated by a simplex σ^0 and constant vector fields $v_1, \ldots, v_j$, with $|v_i| \leqslant |v_j| \leqslant 1$, $1 \leqslant i \leqslant j$, define its *$j$-dipole mass*

$$\|\sigma^j\|_j = |\sigma^0|_0|v_1|\ldots|v_j|.$$

For example, suppose σ^1 is a one-dimensional one-dipole, forming the oppositely oriented sides of a parallelogram. If each of these sides has length ε and each of the other sides has length δ then the dipole mass $\|\sigma^1\|_1 = \varepsilon\delta$, regardless of the angle formed by the parallelogram. Even if the parallelogram is degenerate, the dipole mass is the same.

For j-dipoles $D^j = \sum_{i=1}^k a_i\sigma_i^j$ define *j-dipole mass* as

$$\|D^j\|_j = \sum_{i=1}^k |a_i|\|\sigma_i^j\|_j.$$

r-norms. Let A be a polyhedral chain in $\mathbb{R}^n$ and $r \in \mathbb{Z}$, $r \geqslant 1$. Define

$$|A|_r = \inf \left\{ \sum_{s=0}^{r} \|D^s\|_s + |C|_{r-1} \right\} \tag{1}$$

where the infimum is taken over all dipole decompositions

$$A = \sum_{s=0}^{r} [D^s] + \partial C.$$

These norms in this form were introduced in [H3], although earlier versions appeared in [H1]. We denote the Banach space of p-dimensional polyhedral chains completed with the r-norm by $\mathcal{A}_p^r$.

Lemma 1.1. *If A is polyhedral, $|A|_s \leqslant |A|_r$ for all $r \leqslant s$.*

Proof. This follows directly from the definitions of the norms. □

In [H6] the author shows that the s-norm of a chainlet in $\mathcal{A}_p^r$ is well-defined by taking suitable limits of polyhedral chains and lower semi-continuous. This implies that the Banach spaces $\mathcal{A}_p^r$ are nested and become larger and larger, including more and more strange and pathological elements as r increases. For example, we see in section 2 that the Dirac delta function is represented by a chainlet in $\mathcal{A}_1^1$, its rth derivative (in the sense of distributions) by a chainlet in $\mathcal{A}_1^{r+1}$.

For $r \in \mathbb{Z}^+$, let $\mathcal{B}_p^{r,Lip}$ denote the real linear space of p-forms in $\mathbb{R}^n$ with bounded norm $\|\omega\|_{C^{r,Lip}}$. That is, the r derivatives of each component function of ω exist, have uniformly bounded sup norm and satisfy a uniform Lipschitz condition.

The norms defined here have fractional counterparts [H2, H6] that lead to a definition of a fractal dimension which is more naturally tied to classical theorems of calculus than are other definitions of dimension.

Integration over chainlets

Theorem 1.2. *For A a polyhedral p-chain in $\mathbb{R}^n$ and $\omega \in \mathcal{B}_p^{r-1,Lip}$ then*

$$\left| \int_A \omega \right| \leqslant \|\omega\|_{C^{r-1,Lip}} |A|_r.$$

This is proved in [H5]. (See also [H2].)

The integral of a form $\omega \in \mathcal{B}_p^{r-1,Lip}$ over a chainlet in $A \in \mathcal{A}_p^r$ is defined by taking limits. If $A_k \to A$ are polyhedral chains in $\mathbb{R}^n$ converging to A in the r-norm, define

$$\int_A \omega = \lim_{k\to\infty} \int_{A_k} \omega.$$

This is well defined because of theorem 1.2. This is equivalent† to the alternative definition given in the introduction. If $A = \sum_{i=1}^{\infty} a_i \sigma_i$ then

$$\int_A \omega = \sum_{i=1}^{\infty} \int_{a_i \sigma_i} \omega.$$

† It is worth noting that if an infinite series is conditionally convergent w.r.t. a given norm then the sequence of partial sums converges w.r.t. the norm. Conversely, if $x_n \to x$ is a sequence converging w.r.t. a norm, then the infinite series $x_0 + \sum_{k=1}^{\infty} x_k - x_{k-1}$ conditionally converges w.r.t. the norm.

Another consequence of theorem 1.2 is that $|\quad|_r$ is a norm. Suppose $A \neq 0$ is a polyhedral chain. Then there exists an ∞-smooth form ω such that $\int_A \omega \neq 0$. Thus

$$0 < \left|\int_A \omega\right| \leqslant (p+1)\|\omega\|_{C^{r-1,Lip}}|A|_r.$$

Hence $|A|_r \neq 0$. The other properties of a norm are immediate.

2. Examples

We have chosen four far-ranging examples to illustrate chainlets.

Support of a chainlet. The *support* of a polyhedral chain A is a closed set $spt(A)$ defined as follows: $x \in \mathbb{R}^n \backslash spt(A)$ iff there exists a neighbourhood U of x in $\mathbb{R}^n$ such that if ω is any smooth form supported in U then $\int_A \omega = 0$. The *support* $spt(A)$ of a chainlet $A \in \mathcal{A}_p^r$ is the set of points $q \in \mathbb{R}^n$ such that for every $\varepsilon > 0$ there exists a differential form $\omega \in \mathcal{B}^r$ such that $\int_A \omega \neq 0$ and $\omega(p) = 0$ outside $B_\varepsilon(q)$, the ball of radius ε about q.

It is important to keep in mind that there is much more to a chainlet than the subset of $\mathbb{R}^n$ that forms its support. We will see that there may be many chainlets supported in a given set. For a trivial example, consider a positively oriented two simplex σ in $\mathbb{R}^2$. The chains $\lambda\sigma, \lambda \in \mathbb{R}$, are distinct chains, with the same support. A more interesting example is the solenoid, seen below, which naturally supports quite different chainlets.

1. *Van Koch snowflake.* One may write the snowflake arc S as a sum of simplicial chains $\sum_{k=0}^{\infty} S_k$ where for $k \geqslant 1$, S_k is the sum of 4^k boundaries of triangles σ_k each of side length 3^{-k}. We show this series converges w.r.t. to the one-norm. The partial sums satisfy $S_k + \cdots + S_j = \partial(\sigma_k + \cdots + \sigma_j)$. Thus

$$|S_k + \cdots + S_j|_1 \leqslant |\sigma_k|_0 + \cdots + |\sigma_j|_0 < 4^k/3^{2k}.$$

   Since the rhs tends to 0 as $k, j \to \infty$, we know the infinite sum S is a well-defined chainlet. We conclude that the snowflake is a current and we may integrate Lipschitz differential forms over it. (See figure 2.)

2. *Dirac delta function and its derivatives.* We work in dimension one for simplicity of notation, but the construction can be extended to any dimension. Fix $p \in \mathbb{R}^1$. For each $k \geqslant 0$, let Q_k be a positively oriented interval with length 2^{-k} and centred at p. We claim that the sequence of polyhedral chains $D_k = 2^k Q_k$ converges w.r.t. the one-norm. Notice that the mass of each chain is one. It suffices to estimate $|D_k - D_{k+1}|_1$ We show the difference $D_k - D_{k+1}$ is a one-dimensional one-dipole, a sum of four weighted simple dipoles. Divide D_k into two intervals Q_k with disjoint interiors, of length $2^{-(k+1)}$ and weighted by 2^k. Now D_{k+1} can also be written as the sum of two intervals P_k of length $2^{-(k+1)}$ and weight 2^k, but the line segments are identical to each other. Since the distance between the line segments of P_k and those of Q_k is less than 2^{-k} we deduce

$$|D_k - D_{k+1}|_1 \leqslant 2 2^{-k} 2^k 2^{-(k+1)} = 2^{-k}.$$

   We conclude that the sequence D_k is Cauchy in the one-norm and its limit D has support p. The limit is canonically associated to the Dirac delta functions. Since $D \in \mathcal{A}_1^1$, we may integrate smooth one-forms $\phi\, \mathrm{d}x$ over it. Hence

$$\int_D \phi\, \mathrm{d}x = \lim_{k\to\infty} \int_{D_k} \phi\, \mathrm{d}x = \phi(p) = \delta(\phi).$$

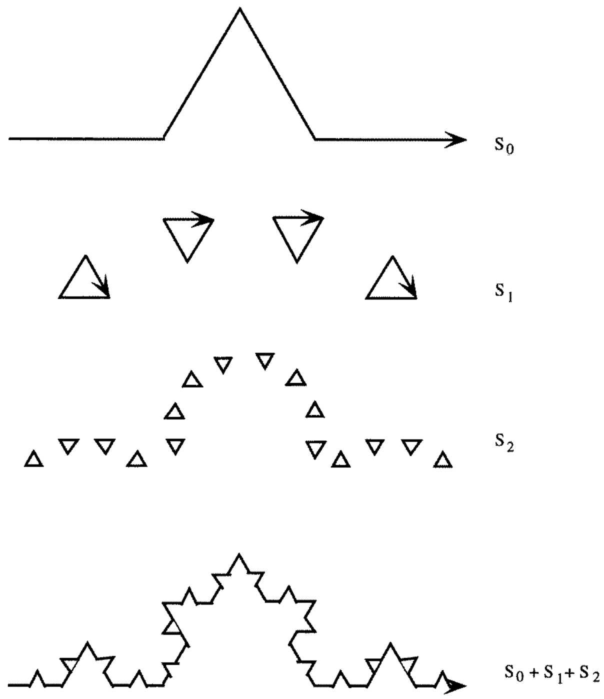

Figure 2. The snowflake as a sum of simplexes

The derivative of the Dirac delta function can also be realized geometrically, but as a chainlet $B \in \mathcal{A}_1^2$. One considers the quadrupoles (or two-dipoles) formed by small oppositely oriented intervals centred at the endpoints of the D_k. It is left as an exercise to show that these two-dipoles limit to a chainlet B and $\int_B \phi \, \mathrm{d}x = \delta'(\phi)$.

3. *Toral solenoid*. Let T be the two-torus in $\mathbb{R}^3$ and $f : T \to T$ a smooth hyperbolic mapping that contracts the torus in one direction, expands it in the other and then wraps the torus around inside itself twice. The solenoid is defined as the intersection $\bigcap_{n=1}^{\infty} f^n T$. It is a set of points that supports many chainlets. For example, let Q be the solid torus positively oriented and $A_0 = Q/|Q|_0$. For $k \geqslant 0$, let $A_{k+1} = f(A_k)/|f(A_k)|_0$. Since the mass stays constant, the analysis here is similar to that for the Dirac delta function and one can use dipoles to show that A_k converges to a nonzero chainlet in $\mathcal{A}_3^1$ with support the solenoid. One can also find chainlets in $\mathcal{A}_1^1$ with support the solenoid as follows. Let B_0 be the oriented core circle in the torus which is not null homotopic. For $k \geqslant 0$, let $B_{k+1} = f(B_k)/|f(B_k)|_0$. Then B_k forms a Cauchy sequence in $\mathcal{A}_1^1$ and thus converges to a chainlet $B \in \mathcal{A}_p^1$. It is also possible to find chainlets in $\mathcal{A}_0^1$ with support the solenoid by choosing a countable dense subset and forming a Dirac mass at each of these points so that their total mass is finite. In the next section we find a chainlet in $\mathcal{A}_2^1$ with support the solenoid.
4. *Graphs of L^1 functions*. The graph Γ of a non-negative L^1 function over an interval $[a, b]$ supports a chainlet. One merely approximates Γ with graphs of monotone increasing step functions Γ_n. These Γ_n are naturally oriented to form simplicial chains and these form a Cauchy sequence in the one-norm. The difference $\Gamma_n - \Gamma_{n+1}$ is a dipole and so its one-norm is bounded above by the area between the two graphs Γ_n and Γ_m. Thus $\{\Gamma_n\}$ forms a Cauchy sequence in the one-norm, converging to a chainlet Γ whose support is

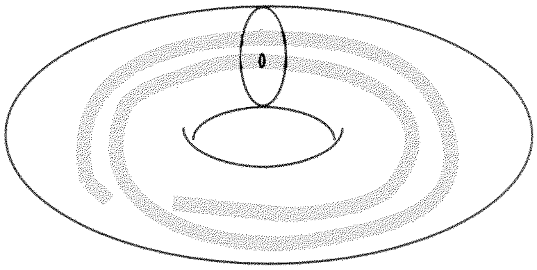

Figure 3. Toral solenoid.

in the graph of f. One can think of Γ as the x-component of the graph of f.

3. Div, grad, and curl for fractals

Banach spaces of chainlets have standard operators defined on them. In this paper we consider the boundary, pushforward, and geometric Hodge * operators. Each is first defined for simplexes and extended to simplicial chains by linearity. Differential forms are used to prove the operators are well defined on polyhedral chains. Finally, the operators are proved to be continuous w.r.t. the norms, showing that they are defined on chainlets. In practice, most of the work comes in establishing the first and last steps. For each operator there is a duality theorem relating chainlets to differential forms. We demonstrate this method of proof for the boundary operator.

3.1. Boundary operator

The boundary of a simplicial chain is defined in the standard way. If $S \sim T$ are simplicial chains we apply Stokes' theorem for simplicial chains to deduce

$$\int_{\partial S} \omega = \int_S \mathrm{d}\omega = \int_T \mathrm{d}\omega = \int_{\partial T} \omega.$$

Hence $\partial S \sim \partial T$, implying that the boundary operator is well defined on polyhedral chains. The boundary operator on polyhedral chains is bounded w.r.t. the r-norms.

Lemma 3.1.

$$|\partial A|_{r+1} \leqslant |A|_r.$$

Proof. This follows immediately from the definition of the r-norms. □

We may thus define the boundary ∂A of a chainlet $A \in \mathcal{A}_p^r$. In particular, the *boundary operator*

$$\partial : \mathcal{A}_p^r \longrightarrow \mathcal{A}_{p-1}^{r+1}$$

is defined for $r \geqslant 0$. It restricts to the usual boundary operator on polyhedral chains and satisfies Stokes' theorem. The boundary operator is dual to the exterior derivative of forms, leading to Stokes' theorem for chainlets.

Theorem 3.2 (Generalized Stokes' theorem). *Let $r \geqslant 0$. If $A \in A_p^r$ and ω is a differential $(p-1)$-form of class $\mathcal{B}^{r+1}$ then*

$$\int_A \mathrm{d}\omega = \int_{\partial A} \omega.$$

Proof. By Stokes' theorem for simplexes, continuity of the boundary operator and of the integral,

$$\begin{aligned}\int_A \mathrm{d}\omega = \int_{\sum_{i=1}^{\infty} a_i\sigma_i} \mathrm{d}\omega = \sum_{i=1}^{\infty} a_i \int_{\sigma_i} \mathrm{d}\omega \quad &= \sum_{i=1}^{\infty} a_i \int_{\partial\sigma_i} \omega \\ &= \int_{\sum_{i=1}^{\infty} a_i \partial\sigma_i} \omega \\ &= \int_{\partial \sum_{i=1}^{\infty} a_i\sigma_i} \omega \\ &= \int_{\partial A} \omega.\end{aligned}$$

□

In [H4] it is shown that this generalization of Stokes' theorem is optimal for integrands of smooth forms. That is, all possible domains D of integration arise as chainlets satisfying continuity: if $\|\omega_k\|_{C^r} \to 0$ then $\int_D \omega_k \to 0$. For one-dimensional chains A Stokes' theorem implies that the fundamental theorem of calculus is valid for all one-dimensional chainlets, e.g., fractal arcs. Here, ω is taken to be a function f and $\mathrm{d}f$ is its gradient. For arcs A with endpoints p and q, this is usually written $\int_p^q f(x)\,\mathrm{d}x = \int_A \mathrm{d}f$.

3.2. Pushforward operator

If f is a mapping of class $\mathcal{B}^{r+1}$, $r \geqslant 0$, one can define the pushforward operator or change of variables operator $f_* : \mathcal{A}_p^r \to \mathcal{A}_p^r$. The pushforward operator on chainlets is dual to the pullback operator on forms leading to a *change of variables* theorem for chainlets.

Theorem 3.3. *Let $r \geqslant 0$. If $A \in \mathcal{A}_p^r$, $\omega \in \mathcal{B}^r$ and $f \in \mathcal{B}^{r+1}$ then*

$$\int_{f_* A} \omega = \int_A f^{\sharp}\omega.$$

(See [H2] for more details.)

Several new operators are defined on chainlets. The main one we discuss here is the geometric Hodge * operator $*$. This, along with the generalized Stokes' theorem, leads to optimal Green and Gauss theorems for chainlets. Combinations and modifications of the operators ∂ and $*$ leads to geometric Laplace operators, Dirac operators, and coboundary operators on chainlets.

3.3. Geometric Hodge star operator

$* : \mathcal{A}_p^r \longrightarrow \mathcal{A}_{n-p}^r$ is defined in [H5] for $r > 0$. To give the idea, figure 3.2 illustrates a polyhedral approximation to $*R$ where R is the oriented rectangle depicted in $\mathbb{R}^3$. $*R$ is found by taking a limit in the one-norm of similar sums of tiny equally spaced one-simplexes, orthogonal to R and whose total length for each sum is the same as the area of R. Even in this simplest example of a rectangle $*R$ is not locally Euclidean, showing that fractal-like structures are naturally associated to smooth ones. (See figure 4.)

Theorem 3.4 (Hodge star theorem). *If $A \in \mathcal{A}_p^r$, and $\omega \in \mathcal{B}^r$ then*

$$\int_A *\omega = \int_{*A} \omega.$$

For the proof, see [H5]. Using a combinatorial definition, others have defined a local dual to the Hodge star operator, but the integral equation of theorem 3.4 does not hold.

The boundary and geometric Hodge star operators lead to generalized Green and Gauss theorems. Let S be a smooth, oriented surface with boundary in $\mathbb{R}^3$. The usual way to integrate

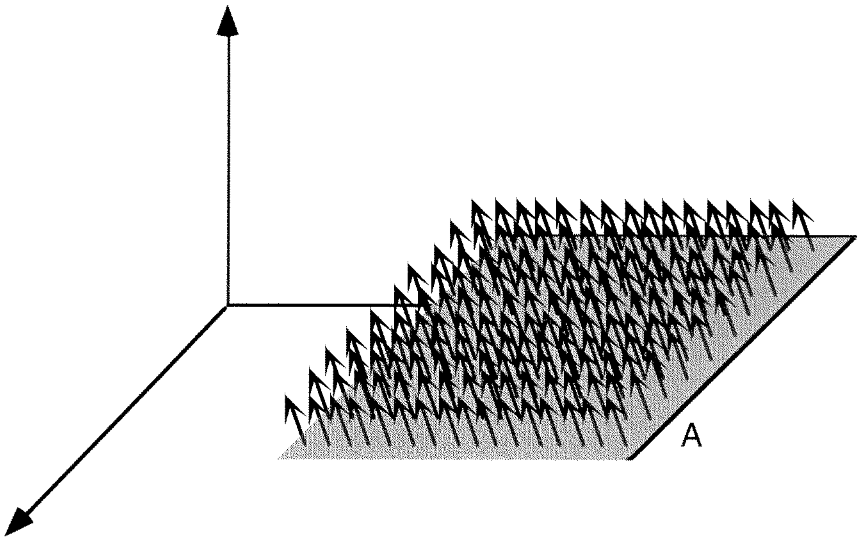

Figure 4. A polyhedral approximation to $*R$

the curl of a vector field X over S is to integrate the dot product of curl$X$ with the unit normal vector field to S. According to Green's theorem, this quantity equals the integral of X over ∂S. We have already seen that we do not require the existence of tangents to ∂S to calculate the integral of X over it. By working with the differential one-form ω associated to X via Euclidean coordinates and the geometric Hodge * operator $*$, we no longer require the existence of any unit normals to S to integrate curl$X$ over A. Instead we use the geometric Hodge star operator applied to A, which is always defined if A is a chainlet. If ω is the differential one-form representing the vector field X it is well known that $*\,\mathrm{d}\omega$ represents curl X. We see now that

$$\int_{*S} *\,\mathrm{d}\omega \text{ represents the integral of the curl}X \text{ over } S.$$

Theorem 3.5 (Fractal Green's theorem). *If S is an chainlet in $\mathcal{A}_p^r$ and $\omega \in \mathcal{B}^{r+1}$ then*

$$\int_{\partial S} \omega = \int_{*S} *\,\mathrm{d}\omega.$$

Proof. By theorems 3.2 and 3.4 $\int_{\partial S} \omega = \int_S \mathrm{d}\omega = \int_{*S} *\,\mathrm{d}\omega$. □

For smooth surfaces with smooth boundary, we have $\int_{\partial S} X \cdot n\mathrm{d}s = \int_{\partial S} \omega$. Since $*\mathrm{d}\omega$ corresponds to curl$X$. It follows that $\int_{*S} *\,\mathrm{d}\omega = \int_S \mathrm{curl}X \cdot n\mathrm{d}A$. Therefore, the preceding theorem generalizes and simplifies Green's theorem: $\int_S \mathrm{curl}X \cdot n\mathrm{d}A = \int_{\partial S} X \cdot n\mathrm{d}s$.

If A is a chainlet then $*\partial A$ plays the role of a normal vector field on its boundary, even though the boundary of A may have no normal vectors defined. Thus one may calculate flux across fractal boundaries and obtain a fractal divergence theorem.

The usual way to calculate the flux of a vector field X across a boundary of a smooth solid region D in space is to integrate the dot product of X with the unit normal vector field to ∂D over the domain ∂D. According to Gauss' divergence theorem, this quantity equals the integral of the divergence of X over D. *By working with the differential form ω associated*

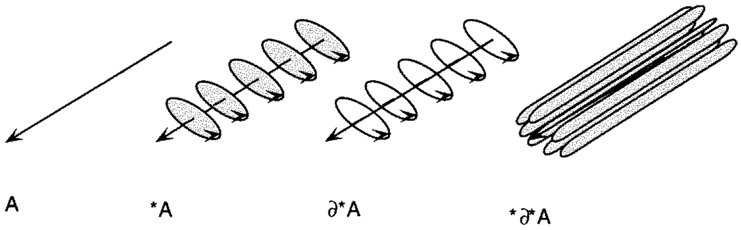

Figure 5. Polyhedral approximation to the coboundary of a line segment.

to X via Euclidean coordinates and the operator $*$, we no longer require the existence of any unit normals to ∂D to calculate the flux of X across ∂D. We see that

$$\int_{*\partial D} \omega \text{ represents the flux of } X \text{ across } \partial D.$$

Theorem 3.6 (Fractal divergence theorem). *If D is a chainlet in $\mathcal{A}_p^r$ and ω is of class $\mathcal{B}^{r+1}$ then*

$$\int_{*\partial D} \omega = \int_D \mathrm{d} * \omega.$$

Proof. $\int_{*\partial D} \omega = \int_{\partial D} *\omega = \int_D \mathrm{d} * \omega$. □

For smooth surfaces, we have $\int_S X \cdot n \mathrm{d}A = \int_{*S} \omega$. It is well known that $\mathrm{d} * \omega$ corresponds to div X. Therefore, the preceding theorem generalizes and simplifies the divergence theorem of Gauss: $\int_{\partial D} X \cdot n \mathrm{d}A = \int_D \mathrm{div} X \mathrm{d}V$.

Some authors have defined the integral over fractal boundaries using the integral of the derived form over the interior, i.e., using the generalized Stokes' theorem, as the definition. Instead, the integrals in the preceding five theorems are defined independently and are shown to satisfy the generalized Stoke's theorem.

3.4. Examples revisited

1. *Van Koch snowflake*. One may calculate flux of a vector field F across the snowflake S as $\int_{*S} \omega$ where ω is the one-form determined by F using the Euclidean inner product.
2. *Dirac delta function and its derivatives*. Distributions and their derivatives can be realized more systematically using the operator $*$ defined in section 3 below. We say a distribution c is *associated* to a chainlet A if $c(\phi) = \int_A \phi \, \mathrm{d}x$ for all test functions ϕ. In [H5] it is shown that if c is a distribution associated to the chainlet A then c' is associated to $*\partial A$.
3. *Toral solenoid*. Recall the chainlet B in $\mathcal{A}_1^1$ found by iterating the core circle via the mapping f. supported in the solenoid. A two-chainlet in $\mathcal{A}_2^1$ can be found by applying the $*$ operator to B. In some real sense, this $*B$ acts as a normal bundle to B.
4. *Graph of an L^1 function*. One may calculate flux of a vector field F across the x-component of the graph Γ of a non-negative L^1 function as $\int_{*\Gamma} \omega$ where, again, ω is the one-form determined by F. We give an important example. Let $F = ye_2$ where $\{e_1, e_2\}$ is the Euclidean basis of $\mathbb{R}^2$. Then $\omega = y\mathrm{d}y$ corresponds to F. Applying Stokes' theorem we calculate the flux of F across Γ to be

$$\int_{*\Gamma} y \, \mathrm{d}y = -\int_\Gamma y \, \mathrm{d}x = \int_S \mathrm{d}x \, \mathrm{d}y = \int_0^1 f(x) \, \mathrm{d}x.$$

Here, S denotes the subgraph of f. This links Lebesgue theory to chainlets.

Geometric Laplace operator. The geometric Hodge * operator $*$ leads immediately to definitions of the geometric coboundary operator δ on chainlets, defined as $\delta = (-1)^{n(p+1)+1} * \partial *$ (see figure 3), and the geometric Laplace operator $\Delta = \delta\partial + \partial\delta$. If $A \in \mathcal{A}_p^r$, then $\delta A \in \mathcal{A}_{p+1}^{r+1}$ and $\Delta A \in \mathcal{A}_p^{r+2}$.

It follows readily from theorems 3.4 and 3.2 that for $\omega \in \mathcal{B}^{r+1}$ and $A \in \mathcal{A}_p^r$

$$\int_A \delta\omega = \int_{\delta A} \omega$$

and for $\omega \in \mathcal{B}^{r+2}$ then

$$\int_A \Delta\omega = \int_{\Delta A} \omega.$$

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